Some Results on the Generalized k-Numerical Range

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ABSTRACT

Let $\mathbf{C}_{n\times n}$ be the linear space of all $n \times n$ complex matrices. Suppose $1 \leqslant k \leqslant n$. The generalized k-numerical range of $A \in \mathbf{C}_{n\times n}$ is defined and denoted by

$$\mathscr{W}_k(A) = \left\{ (x_1^*Ax_1, \dots, x_k^*Ax_k)^t : \{x_1, \dots, x_k\} \text{ is an orthormal family in } \mathbf{C}^n \right\}.$$

When $k = 1$, $\mathscr{W}_k(A)$ is known as the (classical) numerical range (or the field of values) of A, and the concept is well studied. In this note we study the convexity and the geometrical properties of $\mathscr{W}_k(A)$ for general k. In particular, we obtain a necessary and a sufficient condition on A such that $\mathscr{W}_k(A)$ is convex. If the classical numerical range of A equals the convex hull of the spectrum of A, it is shown that the two conditions are equivalent to the convexity of $\mathscr{W}_k(A)$. This result extends that of Poon on this subject. Furthermore, regarding $\mathscr{W}_k(A)$ as a subset of $\mathbf{R}^{2k}$, we give characterizations of scalar matrices, essentially hermitian matrices, and other classes in terms of the affine dimensions of their generalized k-numerical range. We also study the matrices A whose generalized k-numerical range is a polyhedral set.

1. INTRODUCTION

Let $\mathbf{C}_{n\times n}$ be the linear space of all $n \times n$ complex matrices. Suppose $1 \leqslant k \leqslant n$. The *generalized k-numerical range* of $A \in \mathbf{C}_{n\times n}$ is defined and

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denoted by

$$\mathscr{W}_k(A) = \left\{ \left(x_1^* A x_1, \ldots, x_k^* A x_k \right)^t : \{ x_1, \ldots, x_k \} \text{is an orthonormal family in} \, \mathbf{C}^n \right\}.$$

When $k = n$, $\mathscr{W}_k(A)$ is the collection of all diagonal n-tuples of matrices of the form U^*AU where U is unitary. There has been a great deal of interest in studying the diagonal elements of matrices (e.g., see [1, 4, 5, 8, 13, 14, 15]). The motivation comes from both theoretical questions and applied problems. When $k = 1$, $\mathscr{W}_k(A)$ is known as the *(classical) numerical range* (or the *field of values*) of A, and is usually denoted by $W(A)$. The concept is well studied, and many interesting results have been obtained (e.g., see [3, 6] and their references). In particular, a lot of them concern the interesting relations between the geometrical properties of $W(A)$ and the algebraic properties of the matrix A. For example, $W(A)$ is convex for any $A \in \mathbf{C}_{n \times n}$; $W(A)$ is a singleton if and only if A is a scalar matrix; $W(A)$ is a line segment if and only if A is *essentially hermitian*, i.e., $\mu A - \nu I$ is hermitian for some $\mu, \nu \in \mathbf{C}$ with $\mu \neq 0$; $W(A)$ equals the convex hull of $\Lambda(A)$, the spectrum or the set of eigenvalues of A, if A is normal.

The purpose of this note is to study the convexity and some geometrical properties of $\mathscr{W}_k(A)$ for general k. As mentioned above, $\mathscr{W}_1(A) = W(A)$ is always convex. By some results of Horn [8] and Fan and Pall [4], $\mathscr{W}_k(A)$ is convex if A is hermitian. For normal matrices A, Au-Yeung and Sing [1] proved that $\mathscr{W}_n(A)$ is convex if and only if A is essentially hermitian; and Poon [13] gave a necessary and sufficient condition for $\mathscr{W}_k(A)$ to be convex for $1 < k < n$. In [15] Tsing proved that $\mathscr{W}_k(A)$ is always star-shaped with

$$\frac{\operatorname{tr} A}{n} (1, \ldots, 1)^t \in \mathbf{C}^k$$

as a star center for any matrix A and any $k > 1$.

It seems that not much is known about the convexity of $\mathscr{W}_k(A)$ for general matrices A when $k > 1$. In Section 2 we obtain a necessary and a sufficient condition on A such that $\mathscr{W}_k(A)$ is convex. If A satisfies $W(A) = \operatorname{co} \Lambda(A)$, where "co" denotes "the convex hull of," then the two conditions are equivalent to the convexity of $\mathscr{W}_k(A)$. This result extends that of Poon [13]. In Section 3 we study some geometrical properties of the set $\mathscr{W}_k(A)$. Regarding $\mathscr{W}_k(A)$ as a subset of $\mathbf{R}^{2k}$, we give characterizations of scalar matrices, essentially hermitian matrices, etc. in terms of the affine dimensions of their generalized k-numerical range. We also study the matrices A whose generalized k-numerical range is a polyhedral set.

2. CONVEXITY

Since $W(A)$ is always convex, we shall assume $k > 1$ throughout this section. We first establish a sufficient condition on A such that $\mathscr{W}_k(A)$ is convex.

THEOREM 2.1. *Let $k > 1$. Suppose A is unitarily similar to a matrix of the form*

$$A_1 \oplus \cdots \oplus A_k \oplus B$$

such that $W(A) = W(A_1) = \cdots = W(A_k)$. Then $\mathscr{W}_k(A)$ is convex.

Proof. Suppose A satisfies the hypothesis of the theorem. Since $\mathscr{W}_k(A) = \mathscr{W}_k(U^*AU)$ for any unitary matrix U (see [13, Lemma 2]), we may simply assume $A = A_1 \oplus \cdots \oplus A_k \oplus B$. Let $u = (u_1, \dots, u_k)^t$, $v = (v_1, \dots, v_k)^t \in \mathscr{W}_k(A)$. Then $u_i, v_i \in W(A)$ for $i = 1, \dots, k$. By the convexity of $W(A)$, for any $0 \leqslant \lambda \leqslant 1$ we have $w_j = \lambda u_j + (1 - \lambda)v_j \in W(A) = W(A_j)$ for $j = 1, \dots, k$. Thus for $j = 1, \dots, k$, we can find a unit vector $x_j = (x_1^{(j)}, \dots, x_k^{(j)}, x_{k+1}^{(j)})^t \in \mathbf{C}^n$ partitioned according to A with $x_i^{(j)} = 0$ if $i \neq j$ such that $x_j^* A x_j = w_j$. It follows that $w = \lambda u + (1 - \lambda)v = (x_1^* A x_1, \dots, x_k^* A x_k) \in \mathscr{W}_k(A)$. Thus $\mathscr{W}_k(A)$ is convex. ■

The following example communicated by Y. H. Au-Yeung and Y. T. Poon shows that the converse of Theorem 2.1 is not true. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [0] \in \mathbf{C}_{3 \times 3}.$$

Then it can be verified that $\mathscr{W}_k(A)$ is convex for $k = 2, 3$, but A does not satisfy the hypothesis of Theorem 2.1 in either the case of $k = 2$ or $k = 3$.

Next we obtain a necessary condition on A such that $\mathscr{W}_k(A)$ is convex.

THEOREM 2.2. *Let $k > 1$. Suppose A is not essentially hermitian and $\mathscr{W}_k(A)$ is convex. If λ is a nondifferentiable boundary point of $W(A)$, then A is unitarily similar to a matrix of the form $\lambda I_s \oplus B$ with $s \geqslant k$ and $\lambda \notin W(B)$.*

Proof. Suppose A and λ satisfy the hypotheses of the theorem. It is known (e.g., see [3, 12]) that A is unitarily similar to $\lambda I_s \oplus B$ where $\lambda \notin W(B)$. It remains to prove that $s \geqslant k$. Suppose $s < k$. We shall prove that $\mathscr{W}_{s+1}(A)$ is not convex and hence (see [13, Lemma 1]) $\mathscr{W}_k(A)$ is not convex.

Since A is not essentially hermitian, $W(A)$ is not a line segment (see [11, Theorem 2.7]). By the assumption on A, we have (e.g., see [3, 6]) $W(A) = \operatorname{co}[\{\lambda\} \cup W(B)]$. Thus the boundary of $W(A)$ has two edges, say E_1 and E_2, with λ as one of the endpoints. Let γ_i be the point in $W(B) \cap E_i$ which is nearest to λ for $i = 1, 2$. Then we have

$$u = (\underbrace{\lambda, \dots, \lambda}_{s}, \gamma_1)^t,\ v = (\underbrace{\lambda, \dots, \lambda}_{s-1}, \gamma_2, \lambda)^t \in \mathscr{W}_{s+1}(A).$$

We show that $w = (u+v)/2 \notin \mathscr{W}_{s+1}(A)$ as follows. Suppose $\{x_1, \dots, x_{s+1}\}$ is an orthonormal family in $\mathbf{C}^n$ such that $w = (x_1^* A x_1, \dots, x_{s+1}^* A x_{s+1})^t$. Then x_i $(1 \leqslant i \leqslant s-1)$ are eigenvectors of A corresponding to λ. Let x be a unit eigenvector of λ orthogonal to $x_1, \dots, x_{s-1}$, and let U be a unitary matrix whose first s columns are $x_1, \dots, x_{s-1}, x$, respectively. Then $U^*AU = \lambda I_s \oplus C$, where C is unitarily similar to B. Since $\{x_1, \dots, x_{s+1}\}$ is an orthonormal family, we can find orthonormal vectors $y_1, y_2 \in \mathbf{C}^n$ of the form

$$y_i = (\underbrace{0, \dots, 0}_{s-1}, y_s^{(i)}, y_{s+1}^{(i)}, \dots, y_n^{(i)})^t \qquad (i = 1, 2)$$

such that $x_s = Uy_1$ and $x_{s+1} = Uy_2$. As a result, we have

$$\frac{\lambda + \gamma_i}{2} = y_i^* U^* A U y_i = |y_s^{(i)}|^2 \lambda + z^*_i C z_i,$$

where $z_i = (y_{s+1}^{(i)}, \dots, y_n^{(i)})^t$ for $i = 1, 2$. It follows (e.g., see [13, Lemmas 3, 4]) that $|y_s^{(i)}|^2 \geqslant \frac{1}{2}$ for $i = 1, 2$. Since $\{y_1, y_2\}$ is an orthonormal family, we have $y_1^* y_2 = 0$ and thus

$$\begin{aligned} \tfrac{1}{2} \leqslant |y_s^{(1)} y_s^{(2)}| &= |z_1^* z_2| \leqslant \|z_1\| \, \|z_2\| \\ &= \left(1 - |y_s^{(1)}|^2\right)^{1/2} \left(1 - |y_s^{(2)}|^2\right)^{1/2} \leqslant \tfrac{1}{2}. \end{aligned}$$

As a result, $|y_s^{(i)}| = 1/\sqrt{2}$ for $i = 1, 2$, and by the Cauchy-Schwarz inequality the vectors z_1, z_2 are unit multiples of each other. Thus

$$\begin{aligned} \frac{\lambda + \gamma_1}{2} &= x_s^* A x_s = \frac{\lambda}{2} + z_1^* C z_1 \\ &= \frac{\lambda}{2} + z_2^* C z_2 = x_{s+1}^* A x_{s+1} = \frac{\lambda + \gamma_2}{2}, \end{aligned}$$

which is a contradiction. ■

Of course, Theorem 2.2 is not véry useful if $W(A)$ has no nondifferentiable boundary point. On the other hand, if the boundary of $W(A)$ is a nondegenerate convex polygon (which is equivalent to the condition that $W(A) = \operatorname{co} \Lambda(A)$; e.g., see [3, 9, 11]), then we can say more, as shown in Theorem 2.3. In particular, the condition $W(A) = \operatorname{co} \Lambda(A)$ is met when A is normal. Thus Theorem 2.3 can be regarded as an extension of the result of Poon [13].

THEOREM 2.3. *Let $k > 1$. Suppose $A \in \mathbf{C}_{n\times n}$ and the boundary of $W(A)$ is a nondegenerate convex polygon. Then the following conditions are equivalent:*

(a) *$\mathscr{W}_k(A)$ is convex.*

(b) If λ is a vertex of the boundary of W(A), then A is unitarily similar to a matrix of the form $\lambda I_s \oplus B$, where $s \geqslant k$ and $\lambda \notin W(B)$.

(c) *A is unitarily similar to a matrix of the form*

$$\underbrace{A_1 \oplus \cdots \oplus A_1}_{k} \oplus C$$

such that $W(A) = W(A_1)$.

Proof. (a) $\Rightarrow$ (b): Since every vertex of the boundary of $W(A)$ is a nondifferentiable boundary point, the result follows from Theorem 2.2.

(b) $\Rightarrow$ (c): Let $\lambda_1, \dots, \lambda_t$ be the vertices of the boundary of $W(A)$. Then by condition (b), A is unitarily similar to a matrix of the form as described in (c) with $A_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_t)$.

(c) $\Rightarrow$ (a): By Theorem 2.1. ∎

3. SOME GEOMETRICAL PROPERTIES

Since for a given matrix A, the vector $(u_1, \dots, u_{n-1})^t \in \mathscr{W}_{n-1}(A)$ if and only if the vector $(u_1, \dots, u_{n-1}, u_n)^t \in \mathscr{W}_n(A)$ with $u_n = \operatorname{tr} A - \sum_{i=1}^{n-1} u_i$, the geometrical properties of $\mathscr{W}_n(A)$ can be easily deduced from those of $\mathscr{W}_{n-1}(A)$. So we confine our attention to the cases for $k < n$ in the following result.

THEOREM 3.1. *Let $A \in \mathbf{C}_{n\times n}$ and $k < n$. Then exactly one of the following conditions holds:*

(a) *$\mathscr{W}_k(A)$ has real affine dimension 0, and A is a scalar matrix.*

(b) *$\mathscr{W}_k(A)$ has real affine dimension k, and A is a nonscalar essentially hermitian matrix.*

(c) *$\mathscr{W}_k(A)$ has real affine dimension $2k$, and A is not essentially hermitian.*

Proof. Suppose A is a scalar matrix. Then $\mathscr{W}_k(A)$ is a singleton and hence has real affine dimension 0.

Suppose A is a nonscalar essentially hermitian matrix. Then $H = \mu A - \nu I$ is a nonscalar hermitian matrix for some $\mu, \nu \in \mathbf{C}$. Notice that the real affine dimension of $\mathscr{W}_k(A)$ is the same as that of $\mathscr{W}_k(H) \subset \mathbf{R}^k$. The real affine dimension of $\mathscr{W}_k(A)$ is not greater than k. Next we show that there does not exist $\nu \in \mathbf{R}^k$ and $\mu \in \mathbf{R}$ such that

$$\langle u, v\rangle = \mu \qquad \text{for all} \quad u \in \mathscr{W}_k(H).$$

Condition (b) will then follow. Notice that for $v = (v_1, \dots, v_k)^t \in \mathbf{R}^k$, the set

$$S = \{\langle u, v\rangle : u \in \mathscr{W}_k(A)\}$$

can be regarded as the *C-numerical range* of H with $C = \operatorname{diag}(v_1, \dots, v_k, 0, \dots, 0) \in \mathbf{C}_{n \times n}$ (e.g., see [2, 7, 10–12] and their references for definitions and properties of this concept). Since neither H nor C is a scalar matrix, the set S cannot be a singleton (see [11, Theorem 2.5]). Thus our claim is proved.

Finally, suppose A is not essentially hermitian. Since $\mathscr{W}_k(A) \subset \mathbf{C}^k$, the real affine dimension of $\mathscr{W}_k(A)$ is not greater than $2k$. Next we show that there does not exist $v \in \mathbf{C}^k$ and $\mu \in \mathbf{R}$ such that

$$\operatorname{Re}\langle u, v\rangle = \mu \qquad \text{for all} \quad u \in \mathscr{W}_k(A).$$

Condition (c) will then follow. Notice that for $v = (v_1, \dots, v_k)^t \in \mathbf{C}^k$, the set

$$S = \{\langle u, v\rangle : u \in \mathscr{W}_k(A)\}$$

can be regarded as the C-numerical range of A with $C = \operatorname{diag}(\bar{v}_1, \dots, \bar{v}_k, 0, \dots, 0) \in \mathbf{C}_{n \times n}$. Since A is not essentially hermitian, the set S cannot be a line segment in $\mathbf{C}$ (see [11, Theorem 2.7]). Thus the elements in it cannot have the same real part. The result follows. ∎

Notice that when $k = 1$, Theorem 3.1 yields the classical results that $W(A)$ is a singleton if and only if A is a scalar matrix; $W(A)$ is a line segment if and only if A is essentially hermitian.

We shall extend some other results on classical numerical range. To state and prove them we need more definitions and notation.

Let $S \subset \mathbf{R}^m$. A point $u \in S$ is a *conical point* of S if $S \subset u + K$ for some closed convex cone $K \subset \mathbf{R}^m$ that satisfies $K \cap -K = 0$. The set S is *polyhedral* if it is the convex hull of finitely many points in $\mathbf{R}^m$. Suppose $A \in \mathbf{C}_{n \times n}$

has eigenvalues $\lambda_1, \dots, \lambda_n$; we let $\Lambda_k(A)$ be the collection of all the points of the form $(\lambda_{i_1}, \dots, \lambda_{i_k})$, where $i_1, \dots, i_k$ are distinct integers chosen from the set $\{1, \dots, n\}$. Notice that $\Lambda_1(A)$ reduces to $\Lambda(A)$ and we always have $\Lambda_k(A) \subset \mathscr{W}_k(A)$.

The following result can be easily verified (cf. [1]).

THEOREM 3.2. *Let $1 \leqslant k \leqslant n$. If $A \in \mathbf{C}_{n \times n}$ is hermitian, then $\mathscr{W}_k(A) = \operatorname{co} \Lambda_k(A)$. If $A \in \mathbf{C}_{n \times n}$ is normal, then $\operatorname{co} \mathscr{W}_k(A) = \operatorname{co} \Lambda_k(A)$.*

Notice that if $k = 1$, then Theorem 3.2 reduces to the classical result that $W(A) = \operatorname{co} \Lambda(A)$ if A is normal. In fact (see [9, 11]), the following conditions on a matrix A are equivalent:

(a) $W(A) = \operatorname{co} \Lambda(A)$.

(b) $W(A)$ is a polyhedral set.

(c) The matrix A is unitarily similar to $A_1 \oplus A_2$ such that A_1 is normal and $W(A) = W(A_1)$.

This result will be extended in Theorem 3.4. We first prove the following theorem (see [3] for the case of $k = 1$).

THEOREM 3.3. *Let $1 \leqslant k \leqslant n$. Suppose $u = (u_1, \dots, u_k)^t$ is a conical point of $\mathscr{W}_k(A)$. Then A is unitarily similar to a matrix of the form $\operatorname{diag}(u_1, \dots, u_k) \oplus B$. In particular, $u \in \Lambda_k(A)$.*

Proof. Suppose u is a conical point of $\mathscr{W}_k(A)$. Then there exists a closed convex cone K with $K \cap -K = 0$ such that $\operatorname{co} \mathscr{W}(A) \subset u + K$. Then the set of exterior normals to support planes to the set $u + K$ at the point u contains a convex cone with nonempty interior. Thus we can find a vector $v = (v_1, \dots, v_k)^t$ in it such that v has distinct entries and the set

$$S = \{\langle w, v\rangle : w \in u + K\}$$

has a conical point at $\langle u, v\rangle$. Thus $S' = \{\langle w, v\rangle : w \in \mathscr{W}_k(A)\}$ also has a conical point at $\langle u, v\rangle$. Notice that the set S' can be regarded as the C-numerical range of A with $C = \operatorname{diag}(\bar{v}_1, \dots, \bar{v}_k, 0, \dots, 0) \in \mathbf{C}_{n \times n}$. Assume that U is unitary such that the first k diagonal elements of U^*AU equal $u_1, \dots, u_k$. Then $\operatorname{tr}(CU^*AU)$ is a conical point of S'. Thus (see [2, Theorem 3]) $U^*AU = \operatorname{diag}(u_1, \dots, u_k) \oplus B$. ∎

Notice that by Theorem 3.3, if $k \geqslant n - 1$ and $\mathscr{W}_k(A)$ has a conical point, then A is normal.

We are now ready to prove the promised extension of the result stated after Theorem 3.2.

THEOREM 3.4. *Let $1 \leqslant k \leqslant n$ and $A \in \mathbf{C}_{n\times n}$. The following conditions are equivalent*:

(a) $\operatorname{co} \mathscr{W}_k(A) = \operatorname{co} \Lambda_k(A)$.

(b) $\operatorname{co} \mathscr{W}_k(A)$ *is a polyhedral set.*

(c) *The matrix A is unitarily similar to $A_1 \oplus A_2$ such that A_1 is normal and* $\operatorname{co} \mathscr{W}_k(A) = \operatorname{co} \mathscr{W}_k(A_1)$.

Proof. (a) $\Rightarrow$ (b): Clear.

(b) $\Rightarrow$ (c): Suppose condition (b) holds. Let $\lambda_1, \dots, \lambda_t \in \Lambda(A)$ be such that λ_i is an entry of u for some conical point u of $\operatorname{co} \mathscr{W}_k(A)$. Then by Theorem 3.3, A is unitarily similar to $A_1 \oplus A_2$ with $A_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_t)$. Clearly, A_1 is normal and

$$\operatorname{co} \mathscr{W}_k(A) = \operatorname{co} \Lambda_k(A) = \operatorname{co} \Lambda_k(A_1) = \operatorname{co} \mathscr{W}_k(A_1).$$

(c) $\Rightarrow$ (a): Suppose condition (c) holds. Then

$$\operatorname{co} \Lambda_k(A) \subset \operatorname{co} \mathscr{W}_k(A) = \operatorname{co} \mathscr{W}_k(A_1) = \operatorname{co} \Lambda_k(A_1) \subset \operatorname{co} \Lambda_k(A). \qquad \blacksquare$$

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